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Stationary states in a system of two linearly coupled 2D NLS equations with nonlinearities of opposite signs

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Abstract

We study, analytically and numerically, the stationary states in the system of two linearly coupled nonlinear Schrödinger equations in two spatial dimensions, with the nonlinear interaction coefficients of opposite signs. This system is the two-dimensional analogue of the coupled-mode equations for a condensate in the double-well trap (2004 *Phys. Rev.* A **69** 033609). In contrast to the one-dimensional case, where there are stable bright solitons, in two spatial dimensions the Townes-type solitons of the system are unstable. With the use of a parabolic potential, the ground state of the system can be stabilized. It corresponds to strongly coupled condensates and is stable with respect to collapse. This is in sharp contrast to the one-dimensional case, where the ground state corresponds to weakly coupled condensates and is unstable. Moreover, the total number of atoms of the stable ground state can be much higher than the collapse threshold for a single two-dimensional condensate with a negative s-wave scattering length.

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1. Introduction

Bose–Einstein condensates (BECs) in trapped dilute gases exhibit interesting interplay between quantum coherence and nonlinearity since, at zero temperature, the quantum gas is described by the mean-field theory based on the nonlinear Schrödinger equation with an external potential—the Gross–Pitaevskii (GP) equation [1] for the order parameter. The macroscopic quantum coherence of BEC, first experimentally demonstrated in [2, 3], was subsequently explained theoretically [4] with the use of the GP equation. Mean-field theory description, however, neglects the quantum fluctuations, in particular fluctuations of the number of BEC atoms.

Nonlinear phenomena in BEC bear similarity to nonlinear optics. Similar to optics, where the bright and dark solitons are supported, respectively, by the focusing and defocusing nonlinearities, in BECs the s-wave scattering length is the determining factor. Dark solitons are routinely observed in the quasi-one-dimensional condensates with repulsive interactions [5–8]. On the other hand, the attractive one-dimensional BEC propagates in the form of the bright solitons [9, 10].

The s-wave scattering length can be modified by application of a magnetic field near the Feshbach resonance [11]. For a BEC constrained to lower spatial dimensions, the Feshbach resonance still proves to be sharp, for instance, in the two-dimensional condensate [12]. The feasibility of control over the scattering length in BEC by optical means was also proposed [13] (see also [14, 15]).

Control over the scattering length in a part of the condensate can be realized in the double-well trap with far-separated wells. Condensates in the double-well potential are currently routinely created and studied in the experiments (see, for instance, [16–18]).

The double-well trap is created in one spatial dimension, the other two dimensions thus allow for two geometrically distinct cases which correspond to the one- and two-dimensional BECs depending on the trap asymmetry. A combination of the double-well trap with the control over the scattering length allows one to observe two tunnel-coupled BECs with opposite interactions (i.e. one attractive and the other repulsive). In one spatial dimension, such a setup leads to the appearance of the unusual stable bright solitons [19, 20] which have almost all atoms contained in the repulsive condensate. The one-dimensional ground state, however, corresponds to weakly coupled condensates and is unstable with respect to collapse [19]. In this paper, we study the two-dimensional case. We find that the two-dimensional solitons (the Townes-type solitons) are always unstable. However, with the help of a confining potential, the ground state can be stabilized. In contrast to the one-dimensional case, in two dimensions the ground state of the system corresponds to strongly coupled condensates and there is an energy barrier for collapse (see also [21]).

The applicability of the GP-based mean-field theory is limited, but it always applies to the description of stable stationary states. This is due to the similarity between the Bogoliubov–de Gennes equations, describing the surrounding cloud of hot atoms, and the equations describing evolution of a linear perturbation of the order parameter [22]. Thus, a stable mean-field stationary state is also stable in the full quantum approach, its lifetime is equal to that of the condensate. In lower dimensions, the applicability of the mean-field theory is also stipulated by the smallness of the ratio of the average atomic distance to the healing length (see, for instance, [23]).

Theoretical investigations of BEC in the double-well trap go back to the prediction of the anomalous Josephson oscillations [24] and the macroscopic quantum self-trapping of the condensate [25–27].

Experimental advances [16–18] in the production and manipulation of the condensate in the double-well trap make the implementation of the tunnel-coupled repulsive and attractive condensates feasible. The double well is created by focusing an off-resonant laser beam at the centre of the parabolic potential. The minimal separation between the wells is of the order of the corresponding oscillator length of the parabolic potential. In this respect, the optical means for control over the scattering length [13] could be used. Recently, a direct observation of the quantum tunnelling and nonlinear self-trapping of a BEC in the double-well trap was reported [18].

The collapse instability in an attractive BEC loaded into the double-well trap was also recently investigated. In the quasi-one dimensional case, the critical number for collapse turns out to be larger than the same for the corresponding axially symmetric harmonic trap [28]. The tunnelling-induced collapse of an attractive BEC in a double-well trap can take place under the influence of a time-dependent potential [29].

Control over the scattering length in one part of a condensate can be realized in a double-well trap with far-separated wells. In this case, a simplification of the GP equation is possible, which results in a coupled-mode approximation similar to that of [26, 27]. However, in contrast to the latter works, the kinetic energy of the condensate is taken into account (see section 2). The coupled-mode system also has applications in optics (see, for instance, [30–32]).

Since the two-dimensional (2D) nonlinear Schrödinger equation (NLS),

$$\mathrm{i}\partial_t \Psi + \nabla^2 \Psi - g|\Psi|^2\Psi = 0, \tag{1}$$

is critical, we adopt the point of view based on the analysis of the critical scaling and its perturbations. The NLS equation has the following scale invariance: if $\Psi(t, \vec{r})$ is a solution, then $\widetilde{\Psi}(t, \vec{r}) = k\Psi(k^2 t, k\vec{r})$ is also a solution. The number of atoms N (or the l_2-norm), defined by $N = \int \mathrm{d}^n\vec{r}|\Psi|^2$, is scaled as $\tilde{N} = k^{2-n}N$ in n spatial dimensions.

The scale invariance of the 2D NLS equation allows for a family of solutions with the same number of particles. One may call the 2D scale invariance the 'critical scaling'. The critical scaling leads to important consequences (see, for instance, [33]). As the number of particles is constant for the whole family of solutions, the Vakhitov–Kolokolov (VK) criterion [34] applied to the Townes soliton gives marginal stability (or instability), since $\partial N/\partial\mu = 0$, where μ is the chemical potential for a particular solution of the family (i.e. $-\mu$ is the frequency).

This explains why addition of an external *confining* potential allows for stable localized solutions. Indeed, the external potential breaks the scale invariance and the number of particle degeneracy is broken too: some solutions of the former Townes soliton family have number of particles below the collapse threshold (moreover, thanks to the so-called lens transformation [35], the collapse threshold does not depend on the strength of the potential if the latter is parabolic).

Besides addition of an external potential, there are other ways to break the scale invariance. The coupled-mode system, i.e. the system of two linearly coupled NLS equations, provides another way. The coupling coefficient is proportional to the tunnelling rate through the central barrier.

Having understood the relation between the broken scale invariance and stability against collapse, one may wonder if the linear coupling of two 2D NLS equations allows for the existence of stable two-dimensional (i.e. Townes-type) solitons. Indeed, if the critical scaling is broken and the number of atoms (or the number of particles, generally) depends on the chemical potential, then there could be self-localized stationary solutions (which have the right to be called solitons) with $\partial N/\partial\mu < 0$, i.e. satisfying the VK criterion, with the hope that they are stable.

In the following, using the singular perturbation theory supplemented by numerical simulations, we argue that the two-dimensional soliton solutions to the coupled-mode system are always unstable with respect to collapse. The reason is that, in contrast to the 1D case [19], the bifurcation from zero in the 2D coupled-mode system is always discontinuous: the number of atoms of the two-dimensional soliton solution with vanishing amplitude does *not* vanish. However, addition of an external confining potential (a potential transverse to the double-well trap) restores the continuity of the bifurcation from zero and leads to the appearance of stable solutions. Some of them have the number of atoms much larger than the collapse threshold in a single 2D NLS equation. Moreover, such a solution is the ground state of the system, since it has the lowest possible energy for a fixed number of atoms. In contrast to the one-dimensional case, the ground state in the two-dimensional system is secured from collapse by an energy barrier.

The paper is organized as follows. In section 2 we derive the coupled-mode system from the Gross–Pitaevskii equation for a condensate in an asymmetric double-well trap. In section 3 we consider the stability properties of the axially symmetric stationary states. Section 4 is devoted to the study of the limiting case solutions: the bifurcation from zero in the coupled-mode system with and without the parabolic potential, sections 4.1 and 4.2, respectively, and the asymptotic solution corresponding to large negative values of the chemical potential, section 4.3. The reader not interested in details of the bifurcation analysis may go directly to section 5, where we summarize the results and discuss the numerical solution of the coupled-mode system. In the numerics, we have used the Fourier spectral collocation method and looked for stationary solutions using the numerical schemes of [20, 36]. Section 6 contains some concluding remarks.

2. Derivation of the coupled-mode system

The coupled-mode system follows from the GP equation under two conditions. First, the wells of the double-well trap must be far separated. Second, the number of BEC atoms must be below a certain threshold (see equation (11)) with the result that the motion of the condensate in the spatial dimension of the double-well trap is equivalent to that of a quantum particle. The stationary states considered below satisfy this condition. It follows that BEC atoms occupy only the degenerate energy levels of the double-well trap.

The tunnelling coefficient, usually defined through an integral over the overlap of the wavefunctions (see, for instance, [26, 27]), can be, in fact, explicitly given in terms of the parameters of the double-well trap (equation (8)). This is due to the above conditions and the simple fact that the wavefunctions of the localized basis (given by equations (5) and (6)) are uniquely defined by the trap.

The two-dimensional coupled-mode system describes the so-called pancake condensates. Although the 1D and 2D coupled-mode systems are similar, there is a difference in the order of magnitude of the respective parameters which is explained below.

The Gross–Pitaevskii equation for the order parameter $\Psi(\vec{r}, t)$ of a BEC in a double-well trap given by a parabolic potential with a Gaussian barrier reads

$$\mathrm{i}\hbar\partial_t\Psi = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V_{\mathrm{ext}}(\vec{r}) + g|\Psi|^2\right\}\Psi, \tag{2}$$

where

$$V_{\mathrm{ext}} = \frac{m}{2}\left(\omega_\perp^2\vec{r}_\perp^{\,2} + \omega_z^2 z^2\right) + V_{\mathrm{B}}\exp\left\{-\frac{(z-z_0)^2}{2\sigma^2}\right\}, \qquad V_{\mathrm{B}} > 0. \tag{3}$$

Here, the parameters z_0 and σ give the position and width of the barrier, respectively. The pancake geometry corresponds to the condition $\gamma \equiv \omega_z/\omega_\perp \gg 1$. In terms of the potential from equation (3), the wells of the double-well trap are far separated if $\sigma \sim \ell_z$, with $\ell_z \equiv \sqrt{\frac{\hbar}{m\omega_z}}$ being the oscillator length in the z-direction.

The motion in the double-well trap and the transverse dynamics of BEC can be factorized if the number of BEC atoms is not large (see equation (11)), i.e. if the oscillator length ℓ_z is much smaller than the characteristic length of nonlinearity ℓ_{nl}. In the case of a double-well with far-separated wells, the first two energy levels are quasi-degenerate:

$$E_1 - E_0 \ll E_2 - E_1. \tag{4}$$

Being interested in the ground state of the system (occupied by the condensate), we consider only the degenerate subspace. This approximation neglects the uncondensed atomic cloud which can be discarded for temperatures below the condensation threshold [1].

The basis in the degenerate subspace for expansion of the order parameter is dictated by the necessity to simplify the nonlinear term, which is not small for the transverse degrees of freedom, i.e. in the pancake plane. Evidently, one should select such a basis, say $\psi_u(z)$ and $\psi_v(z)$, where each basis function is *localized in just one of the wells*. The localized basis is given by a rotation of the wavefunctions for the ground state and the first excited state:

$$\psi_u(z) = \frac{\psi_0(z) + \varkappa\psi_1(z)}{\sqrt{1+\varkappa^2}}, \qquad \psi_v(z) = \frac{\varkappa\psi_0(z) - \psi_1(z)}{\sqrt{1+\varkappa^2}}. \tag{5}$$

Obviously, the new wavefunctions are orthogonal and normalized. The parameter $\varkappa$ is selected by the quotient of the absolute values of the eigenfunctions $\psi_0(z)$ and $\psi_1(z)$ at the minima of the double-well potentials z_- and z_+ (say, $z_- < z_+$) [19]. We set

$$\varkappa = \frac{\psi_1(z_-)}{\psi_0(z_-)} \approx -\frac{\psi_0(z_+)}{\psi_1(z_+)} \tag{6}$$

(the positions of the extremals of the wavefunctions slightly deviate from the minima of the trap; these deviations we neglect). For this choice of $\varkappa$, the wavefunctions $\psi_u(z)$ and $\psi_v(z)$ defined by equation (5) are localized in the left and right wells, respectively (see also figures 1 and 2 in [19]).

The Hamiltonian for a quantum particle in the double-well potential, projected on the degenerate subspace, i.e. $H_z = E_0|\psi_0\rangle\langle\psi_0| + E_1|\psi_1\rangle\langle\psi_1|$, in the new basis becomes

$$H_z = \bar{E}(|\psi_u\rangle\langle\psi_u| + |\psi_v\rangle\langle\psi_v|) + |\psi_v\rangle\mathcal{E}\langle\psi_v| - K(|\psi_u\rangle\langle\psi_v| + |\psi_v\rangle\langle\psi_u|). \tag{7}$$

Here,

$$\bar{E} = \frac{\varkappa^2(E_0+E_1)}{1+\varkappa^2}, \qquad \mathcal{E} = \frac{1-\varkappa^2}{1+\varkappa^2}(E_1 - E_0), \qquad K = \frac{\varkappa(E_1-E_0)}{1+\varkappa^2}. \tag{8}$$

We can set $\bar{E} = 0$ without loss of generality. Equation (8) gives the tunnelling coefficient K and the zero-point energy difference $\mathcal{E}$.

We approximate the solution of the GP equation (2) by a sum of the factorized order parameters:

$$\Psi(t, \vec{r}_\perp, z) = \Phi_u(t, \vec{r}_\perp)\psi_u(z) + \Phi_v(t, \vec{r}_\perp)\psi_v(z). \tag{9}$$

Let us now formulate the condition for the factorization in equation (9). We have neglected the nonlinear term in the Gross–Pitaevskii equation as compared to the longitudinal kinetic term, that is

$$\frac{\hbar^2}{2m}\frac{1}{\ell_z^2} \gg |g||\Phi|^2|\psi|^2 \tag{10}$$

(for each of the two wells). The wavefunctions can be estimated as follows: $|\psi|^2 \sim 1/\ell_z$ and $|\Phi|^2 \sim \mathcal{N}/d_\perp^2$, with $d_\perp$ being the transverse radius of the condensate and $\mathcal{N}$ the number of atoms (in the considered well). Using the expression for the nonlinear coefficient in the Gross–Pitaevskii equation $g = 4\pi\hbar^2 a_s/m$ [1], where a_s is the atomic scattering length, we get the applicability condition for the separation of variables as follows:

$$\frac{8\pi|a_s|\ell_z\mathcal{N}}{d_\perp^2} \ll 1. \tag{11}$$

Condition (11) must be satisfied by both condensates in the double-well trap.

The coupled-mode system is derived by inserting expansion (9) into equation (2), using the orthogonality of the basis functions $\psi_{u,v}$, the formulae

$$H_z\psi_u = -K\psi_v, \qquad H_z\psi_v = \mathcal{E}\psi_v - K\psi_u$$

and throwing away the small nonlinear terms involving the cross-products of the localized wavefunctions ψ_u and ψ_v (see also the discussion of [19]). One arrives at the system

$$\mathrm{i}\hbar\partial_t\Phi_u = -\frac{\hbar^2}{2m}\nabla_\perp^2\Phi_u + V(\vec{r}_\perp)\Phi_u + g_u|\Phi_u|^2\Phi_u - K\Phi_v, \tag{12a}$$

$$\mathrm{i}\hbar\partial_t\Phi_v = -\frac{\hbar^2}{2m}\nabla_\perp^2\Phi_v + V(\vec{r}_\perp)\Phi_v + (\mathcal{E} + g_v|\Phi_v|^2)\Phi_v - K\Phi_u. \tag{12b}$$

Here, $g_{u,v} \equiv \int \mathrm{d}z\, g(z)|\psi_{u,v}|^4$ and $V(\vec{r}_\perp) = \frac{m\omega_\perp^2}{2}\vec{r}_\perp{}^2$ (by changing the sign of either ψ_u or ψ_v, one can always set $K > 0$).

System (12) is the basis of our approach. Conditions (4) and (11) are satisfied by all stable stationary states considered below. In the search for two-dimensional soliton solutions, we will neglect the confining transverse potential $V(\vec{r}_\perp)$ (i.e. when it is considered as flat on the scale of the Townes-like soliton solution). This case will be called below 'the coupled-mode system without the transverse potential'. The atomic interaction in the u-condensate is attractive, $g_u < 0$, while in the v-condensate it is externally modified to repulsive, $g_v > 0$.

For the numerical and analytical analysis, it is convenient to use the dimensionless variables defined as follows:

$$T = \frac{\omega_\perp}{2}t, \qquad \vec{\rho} = \frac{\vec{r}}{\ell_\perp}, \qquad \ell_\perp \equiv \sqrt{\frac{\hbar}{m\omega_\perp}}. \tag{13}$$

The order parameters are expressed as

$$\Phi_u = \frac{\sqrt{\Delta}}{\ell_\perp}u, \qquad \Phi_v = \frac{\sqrt{\Delta}}{\ell_\perp}v, \tag{14}$$

where $\Delta = \left(\sqrt{8\pi|a_s^{(u)}|\int \mathrm{d}z|\psi_u|^4}\right)^{-1}$ and $a_s^{(u)}$ is the scattering length in the u-condensate. The dimensionless system reads

$$\mathrm{i}\partial_T u = \left(-\nabla_{\vec{\rho}}^2 + \vec{\rho}^2\right)u - |u|^2 u - \kappa v, \tag{15a}$$

$$\mathrm{i}\partial_T v = \left(-\nabla_{\vec{\rho}}^2 + \vec{\rho}^2\right)v + (\varepsilon + a|v|^2)v - \kappa u. \tag{15b}$$

Here

$$a = \frac{a_s^{(v)}}{|a_s^{(u)}|}\frac{\int \mathrm{d}z|\psi_v|^4}{\int \mathrm{d}z|\psi_u|^4}, \qquad \kappa = \frac{2K}{\hbar\omega_\perp} = \frac{2\varkappa}{1+\varkappa^2}\frac{E_1 - E_0}{\hbar\omega_\perp}, \qquad \varepsilon = \frac{2\mathcal{E}}{\hbar\omega_\perp} = 2\frac{1-\varkappa^2}{1+\varkappa^2}\frac{E_1 - E_0}{\hbar\omega_\perp}. \tag{16}$$

The number of atoms $\mathcal{N}$ in the condensate (the l_2-norm) is given as follows:

$$\mathcal{N}_{u,v} = \int \mathrm{d}^2\vec{\rho}|\Phi_{u,v}|^2 = \Delta N_{u,v}, \qquad N_u \equiv \int \mathrm{d}^2\vec{\rho}|u|^2, \qquad N_v \equiv \int \mathrm{d}^2\vec{\rho}|v|^2. \tag{17}$$

The quantity $N_{u,v}$ will be referred to as the 'number of atoms' for short, since we are interested only in the relative shares of the number of atoms in the two condensates and the ratio of the total number of atoms to the collapse threshold in a single 2D NLS. The transformation coefficient Δ can be estimated as $\frac{\ell_z}{8\pi|a_s|}$, it is of order 10^2–10^3 for the current trap sizes in the experiments with BECs.

The tunnelling coefficient κ and the zero-point energy difference ε of the dimensionless coupled-mode system can take arbitrary values. Indeed, from the definition (16), we have

$$\kappa = \frac{2K}{\hbar\omega_\perp} = \gamma\frac{2K}{\hbar\omega_z}, \qquad \varepsilon = \frac{2\mathcal{E}}{\hbar\omega_\perp} = \gamma\frac{2\mathcal{E}}{\hbar\omega_z},$$

i.e. there are two unrelated multipliers, the first, γ, is large and the second is small.

Finally, we can reformulate the applicability condition (11) for the coupled-mode system in the dimensionless variables

$$N_u \ll \gamma r_u^2, \qquad a N_v \ll \gamma r_v^2, \tag{18}$$

where $r_{u,v} = d_\perp^{(u,v)}/\ell_\perp$ is the dimensionless radius of the condensate. Condition (18) is derived by using the transformation (17) with the estimate $\int \mathrm{d}z|\psi_{u,v}|^4 \sim 1/\ell_z$ and $\gamma = \ell_\perp^2/\ell_z^2$.

We have verified that condition (18) is satisfied (for the pancake trap with $\gamma \geqslant 100$) by the stable stationary states (figures 3, 4, 6 and 7).

If more than two energy levels of the double-well trap are significantly occupied by the condensate, the conditions for applicability of the coupled-mode system (12) are violated. In this case, one can use the nonlinear coupled-mode approach [37] which results, however, in the nonlinearly coupled NLS equations.

3. Stability of the stationary states

BEC in a double-well trap can be unstable with respect to collapse if the atomic interaction is attractive (the u-condensate in the notation of the previous section). By setting $\kappa = 0$ in system (15), we obtain for the u-condensate the focusing 2D NLS equation with an external potential. In the simplest case when the potential is parabolic, the collapse threshold is given by the l_2-norm of the Townes soliton:

$$N_{\mathrm{th}} = 11.69, \tag{19}$$

since the collapse threshold is independent of the parabolic potential [35, 33]. The Townes soliton is the solution $u = \mathrm{e}^{\mathrm{i}T} R(\rho)$ of the 2D NLS equation, i.e. the function $R(\rho)$ satisfies

$$\nabla^2 R + R^3 - R = 0. \tag{20}$$

Here and below, the operator ∇ is the gradient with respect to $\vec{\rho}$ if it is not explicitly indicated otherwise by a subscript. In the following, we will need the well-known identity for the Townes soliton (see, for instance, [33])

$$\int \mathrm{d}^2\vec{\rho}\, R^2 = \int \mathrm{d}^2\vec{\rho} (\nabla R)^2 = \frac{1}{2}\int \mathrm{d}^2\vec{\rho}\, R^4. \tag{21}$$

Let us now consider the problem of stability of the stationary states in the coupled-mode system. We are interested only in the axially symmetric stationary states, $u = \mathrm{e}^{-\mathrm{i}\mu T} U(\rho)$ and $v = \mathrm{e}^{-\mathrm{i}\mu T} V(\rho)$, where $\rho = |\vec{\rho}|$. The stability or instability can be established by considering the eigenvalue problem associated with the linearized system. Writing the perturbed solution as follows:

$$u = \mathrm{e}^{-\mathrm{i}\mu T}\{U(\rho) + \mathrm{e}^{-\mathrm{i}\Omega T}\mathcal{U}(\vec{\rho})\}, \qquad v = \mathrm{e}^{-\mathrm{i}\mu T}\{V(\rho) + \mathrm{e}^{-\mathrm{i}\Omega T}\mathcal{V}(\vec{\rho})\}, \tag{22}$$

where $(\mathcal{U}, \mathcal{V})$ is a small perturbation mode with the frequency Ω which comprise a solution of the eigenvalue problem:

$$-\mathrm{i}\Omega \begin{pmatrix}\mathcal{U}_R\\ \mathcal{V}_R\end{pmatrix} = \Lambda_0 \begin{pmatrix}\mathcal{U}_I\\ \mathcal{V}_I\end{pmatrix}, \qquad \mathrm{i}\Omega \begin{pmatrix}\mathcal{U}_I\\ \mathcal{V}_I\end{pmatrix} = \Lambda_1 \begin{pmatrix}\mathcal{U}_R\\ \mathcal{V}_R\end{pmatrix}, \tag{23}$$

with

$$\Lambda_0 = \begin{pmatrix} L_0^{(u)} & -\kappa \\ -\kappa & L_0^{(v)} \end{pmatrix}, \qquad \Lambda_1 = \begin{pmatrix} L_1^{(u)} & -\kappa \\ -\kappa & L_1^{(v)} \end{pmatrix}. \tag{24}$$

Here, the scalar operators are defined as follows:

$$L_0^{(u)} = -(\nabla^2 + U^2 + \mu) + \rho^2, \qquad L_1^{(u)} = -(\nabla^2 + 3U^2 + \mu) + \rho^2,$$
$$L_0^{(v)} = -(\nabla^2 - aV^2 + \mu - \varepsilon) + \rho^2, \qquad L_1^{(v)} = -(\nabla^2 - 3aV^2 + \mu - \varepsilon) + \rho^2. \tag{25}$$

First of all, the matrix operator Λ_0 is non-negative for positive stationary solutions, i.e. satisfying $UV > 0$. Indeed, the scalar operators on the main diagonal of Λ_0 can be cast as follows:

$$L_0^{(u)} = -\frac{1}{U}\nabla U^2\nabla\frac{1}{U} + \kappa\frac{V}{U}, \qquad L_0^{(v)} = -\frac{1}{V}\nabla V^2\nabla\frac{1}{V} + \kappa\frac{U}{V},$$

which can be easily verified by direct calculation. Therefore, the scalar product of Λ_0 with any vector $X = (X_1(\vec{\rho}), X_2(\vec{\rho}))$ is non-negative:

$$\langle X|\Lambda_0|X\rangle = \int \mathrm{d}^2\vec{\rho}\left\{X_1^* L_0^{(u)} X_1 + X_2^* L_0^{(v)} X_2 - \kappa(X_1^* X_2 + X_2^* X_1)\right\}$$
$$\geqslant \kappa \int \mathrm{d}^2\vec{\rho}\left|X_1\sqrt{\frac{V}{U}} - X_2\sqrt{\frac{U}{V}}\right|^2 \geqslant 0.$$

Here, we have used the positivity of the operators $-\frac{1}{U}\nabla U^2\nabla\frac{1}{U}$ and $-\frac{1}{V}\nabla V^2\nabla\frac{1}{V}$. The operator Λ_0 has one zero mode given by the stationary point itself: $Z = (U, V)$, $\Lambda_0 Z = 0$.

Non-negativity of Λ_0 is an essential property for the following, therefore we will concentrate on the solutions satisfying $UV > 0$, which can be termed 'positive' solutions, while the ones satisfying $UV < 0$ are discarded from the consideration below. The latter solutions bifurcate from zero at a higher energy than the positive ones (see sections 4.1 and 4.2).

For a positive solution, the lowest eigenfrequency of the linear stability problem can be found by minimizing the quotient

$$\Omega^2 = \min\frac{\langle X|\Lambda_1|X\rangle}{\langle X|\Lambda_0^{-1}|X\rangle} \tag{26}$$

in the space orthogonal to the zero mode of Λ_0: $\langle Z|X\rangle = 0$ (here $\langle X|Y\rangle \equiv \int \mathrm{d}^2\vec{\rho}(X_1^*Y_1 + Y_2X_2^*)$). Equation (26) follows from the eigenvalue problem rewritten as $\Lambda_0\Lambda_1 X = \Omega^2 X$ with $X = (\mathcal{U}_R, \mathcal{V}_R)$.

The imaginary eigenfrequencies Ω, which mean instability, appear due to negative eigenvalues of the operator Λ_1. For the coupled-mode system without the transverse potential, there is at least one zero eigenvalue of Λ_1 due to the translational invariance, with the eigenfunction (in the vector form) being given as $Z_1 = (\nabla U, \nabla V)$. For the positive stationary solutions, if there is only one negative eigenvalue, the VK stability criterion $\frac{\partial N}{\partial\mu} < 0$ applies, which can be established by a simple repetition of the arguments presented, for instance, in [38]. The limit on the number of negative eigenvalues is related to the fact that the minimization of the quotient in equation (26) is subject to only one orthogonality condition, thus only one negative direction in the energy functional can be eliminated by satisfying this condition.

The following simple strategy has been used to establish the stability. The eigenvalue problem was reformulated in the polar coordinates (ρ, θ) and the operators were expanded in Fourier series with respect to θ, which is done by the simple substitution: $\nabla^2 \to \nabla_\rho^2 - n^2/\rho^2$, where $\nabla_\rho^2 = \partial_\rho^2 + \rho^{-1}\partial_\rho$. Noting that the orbital operators $\Lambda_{1,n} = \Lambda_1(\nabla^2 \to \nabla_\rho^2 - n^2/\rho^2)$ are ordered as follows: $\Lambda_{1,n+1} \geqslant \Lambda_{1,n}$, we have checked for the negative eigenvalues of the first two orbital operators with $n = 0, 1$. If there are two or more negative eigenvalues (for instance, if $\Lambda_{1,0}$ and $\Lambda_{1,1}$ both have one negative eigenvalue), then the solution is unstable; if there is only one such $\Lambda_{1,0}$, then one can apply the VK criterion. From our numerical simulations it follows that $\Lambda_{1,1}$ is always positive, while the operator $\Lambda_{1,0}$ has one negative eigenvalue or none (the latter case corresponds to the defocusing effective nonlinearity in the coupled-mode system with the transverse potential, see sections 4.2 and 5).

When the operator Λ_1 does not have negative eigenvalues at all, the solution is unconditionally stable. In this case, the formal threshold $\frac{\partial N}{\partial \mu} = 0$ of the stability criterion is the point where an additional zero mode of the operator $\Lambda_{1,0}$ appears. Indeed, at the threshold point, we have

$$\int \mathrm{d}^2\vec{\rho} \left(\frac{\partial U}{\partial \mu}, \frac{\partial V}{\partial \mu} \right) \Lambda_{1,0} \begin{pmatrix} \frac{\partial U}{\partial \mu} \\ \frac{\partial V}{\partial \mu} \end{pmatrix} = 0,$$

due to the identity

$$\Lambda_{1,0} \begin{pmatrix} \frac{\partial U}{\partial \mu} \\ \frac{\partial V}{\partial \mu} \end{pmatrix} = \begin{pmatrix} U \\ V \end{pmatrix}, \tag{27}$$

which follows from differentiation of the stationary coupled-mode system, $\Lambda_0(U, V)^T = 0$, with respect to μ. Hence, in this case, on one side of the VK threshold the operator $\Lambda_{1,0}$ is positive and the solution is unconditionally stable, while on the other side there is one negative eigenvalue and the VK criterion applies (this corresponds to a change of the effective interaction from the repulsive to attractive, see sections 4.2 and 5).

Note that the above approach allows us to decide on the stability of the stationary solutions to the coupled-mode system without numerical solution of the full eigenvalue problem (23).

4. Small-amplitude and asymptotic solutions

In this section we study two limiting cases of solutions of the coupled-mode system (15): the solution with vanishing amplitude (i.e. the bifurcation from zero) and the asymptotic solution with the chemical potential taking large negative values. There is an essential difference in the bifurcating small-amplitude solutions in the coupled-mode systems with and without the transverse potential, while the asymptotic solution, though also possessing some minor difference between the two cases, can be studied for both cases simultaneously. Accordingly, the section is divided into three subsections. First, we study the soliton bifurcation from zero, i.e. the coupled-mode system without the transverse potential, in section 4.1. In section 4.2, the same bifurcation is considered in the coupled-mode system with a general transverse potential. We derive the asymptotic solution in section 4.3 using, for simplicity, the parabolic transverse potential.

4.1. The soliton bifurcation from zero

Though the linear coupling of the repulsive and attractive NLS equations breaks the scale invariance, the 2D soliton solutions are always unstable. This conclusion follows from a comparison of the soliton bifurcation from zero, i.e. $\mu \to \mu_{\mathrm{bif}}$ (which is the subject of this section) and the asymptotic solution when $\mu \to -\infty$ (which is considered in section 4.3). Moreover, it is supported by the direct numerical solution of section 5.

The stationary coupled-mode system (when the transverse potential is flat) reads

$$\mu U + \nabla^2 U + U^3 + \kappa V = 0, \tag{28a}$$

$$(\mu - \varepsilon) V + \nabla^2 V - a V^3 + \kappa U = 0. \tag{28b}$$

First of all, the most important information on the existence of solitons is provided by the dispersion law of the linearized system, which has two branches in the case of the coupled-mode system (28). Setting $U = U_0\,\mathrm{e}^{-\lambda\rho}$ and $V = V_0\,\mathrm{e}^{-\lambda\rho}$, in the limit of vanishing U_0 and V_0, we obtain

$$\lambda_{1,2}^2 = \mu_{1,2} - \mu, \qquad \mu_1 = \frac{\varepsilon}{2} - \sqrt{\frac{\varepsilon^2}{4} + \kappa^2}, \qquad \mu_2 = \frac{\varepsilon}{2} + \sqrt{\frac{\varepsilon^2}{4} + \kappa^2}. \tag{29}$$

It is easy to see that the following inequalities hold: $\mu_1 < 0 < \mu_2$ and $\mu_1 < \varepsilon < \mu_2$.

The same dispersion relations appear also in the 1D case as well [19], their universality is due to the fact that the limiting values of the chemical potential are determined by the energy difference in the double-well trap and the trap asymmetry:

$$\mu_1 = \left(\frac{1-\varkappa^2}{1+\varkappa^2} - \frac{1}{2}\right)\frac{E_1 - E_0}{\hbar\omega_\perp}, \qquad \mu_2 = \left(\frac{1-\varkappa^2}{1+\varkappa^2} + \frac{1}{2}\right)\frac{E_1 - E_0}{\hbar\omega_\perp}. \tag{30}$$

In the 1D case, the positive solitons ($UV > 0$) bifurcate from zero at the lower energy level $\mu = \mu_1$, while the non-positive ones ($UV < 0$) bifurcate from the higher level $\mu = \mu_2$. This property also holds in the 2D case. This can be shown as follows. First of all, only one branch of soliton solutions may correspond to each branch of the dispersion law. Second, there is a point ρ_0 on the positive real line such that $\nabla^2 U(\rho_0) = 0$. Indeed, setting $\xi = \ln\rho$ we have $\nabla^2 U(\rho) = \mathrm{e}^{-2\xi}\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}$, whereas, for the solution with a finite l_2-norm, the first derivative $\frac{\mathrm{d}U}{\mathrm{d}\xi} = \rho\frac{\mathrm{d}U}{\mathrm{d}\rho}$ tends to zero as $\xi \to \pm\infty$ (i.e. when $\rho \to 0$ or $\rho \to \infty$). Considering equation (28a) at $\rho = \rho_0$, we obtain $(\mu U + U^3 + \kappa V)|_{\rho_0} = 0$, thus $\mu < 0$ for $UV > 0$ (recall that $\kappa > 0$), i.e. the positive solitons belong to the λ_1-branch of the dispersion law, while the non-positive ones belong to the λ_2-branch.

Consider now the positive solitons with vanishing amplitude, i.e. in the limit $\mu \to \mu_{\mathrm{bif}} \equiv \mu_1$. Set $\mu = \mu_1 - \epsilon$, with $\epsilon \to 0$ and suppose that in this limit $U = \mathcal{O}(\epsilon^p)$ and $V = \mathcal{O}(\epsilon^q)$ with some $p, q > 0$. From system (28) we necessarily get $p = q$. Further steps are essentially the same as in the 1D case [19]. First, we expand V in the power series with respect to U and its derivatives, using for this goal equation (28b):

$$V = \kappa\beta U - a\kappa^3\beta^4 U^3 + 3a^2\kappa^5\beta^7 U^5 + \kappa\beta^2\nabla^2 U + \mathcal{O}(\epsilon^s), \tag{31}$$

where $\beta = (\varepsilon - \mu)^{-1}$ and $s = \min\{3p+1, p+2, 7p\}$. Here, β should also be expanded with respect to ϵ. Second, the result is substituted into equation (28a) and we get

$$-\epsilon\left(1 + \frac{\kappa^2}{\mu_2^2}\right)U + \left(1 + \frac{\kappa^2}{\mu_2^2}\right)\nabla^2 U + \left(1 - \frac{a\kappa^4}{\mu_2^4}\right)U^3 + \frac{3a^2\kappa^6}{\mu_2^7}U^5 = \mathcal{O}(\epsilon^s). \tag{32}$$

From the coefficient at the cubic term and the definition of μ_2, one immediately concludes that the soliton bifurcation from zero is possible if and only if $\varepsilon \geqslant \varepsilon_{\mathrm{cr}}$, where $\varepsilon_{\mathrm{cr}}$ is the same as in the 1D case [19]:

$$\varepsilon_{\mathrm{cr}} = \kappa(a^{1/4} - a^{-1/4}). \tag{33}$$

For $\varepsilon > \varepsilon_{\mathrm{cr}}$, one can drop the quintic term from equation (32), thus the effective bifurcation equation is the canonical 2D NLS equation except for the coefficients. In this case, the condition that all terms have the same order requires that $p = 1/2$ and $\nabla^2 \sim \epsilon$, i.e. there is a new length scale $\xi = \sqrt{\epsilon}\rho$. The soliton solution reads

$$U = \sqrt{\epsilon}A_0\{R(\sqrt{\epsilon}\rho) + \epsilon\mathcal{G}(\sqrt{\epsilon}\rho) + \mathcal{O}(\epsilon^2)\}, \qquad A_0 \equiv \left(1 + \frac{\kappa^2}{\mu_2^2}\right)^{1/2}\left(1 - \frac{a\kappa^4}{\mu_2^4}\right)^{-1/2}. \tag{34}$$

Here, $R(\xi)$ is the Townes soliton.

While the soliton amplitude approaches zero, the corresponding number of atoms has a *finite* limit. Indeed, we have

$$N_u = \int \mathrm{d}^2\vec{\rho}\, U^2 = A_0^2 N_{\mathrm{th}} + \mathcal{O}(\epsilon), \tag{35}$$

with N_{th} being the threshold for collapse (19) in a single 2D NLS equation. Thus, the bifurcation from zero is discontinuous in the 2D case. This is quite dissimilar to the 1D case, where only on the boundary $\varepsilon = \varepsilon_{\mathrm{cr}}$ is the soliton bifurcation from zero discontinuous [19].

On the boundary $\varepsilon = \varepsilon_{\mathrm{cr}}$, the effective equation is the quintic NLS equation and $p = 1/4$, exactly as in the 1D case. However, in contrast to the 1D case, the bifurcation from zero is singular. Now $U = \epsilon^{1/4} B_0(F_0(\sqrt{\epsilon}\rho) + \epsilon F_1(\sqrt{\epsilon}\rho) + \mathcal{O}(\epsilon^2))$, with some B_0 and $F_0(\xi)$. Hence, the number of atoms $N_u \propto \epsilon^{-1/2}$ as $\epsilon \to 0$. The bifurcating solitons are unstable in this case by the VK criterion.

For $\varepsilon > \varepsilon_{\mathrm{cr}}$, to decide on the stability of the soliton solutions near the bifurcation point one has to find the derivative of the number of atoms with respect to ϵ in the limit $\epsilon \to 0$. To this end, the soliton solution up to the order $\epsilon^{3/2}$ is required. The necessary higher order expansion for V in terms of U reads

$$V = \kappa\beta U - a\kappa^3\beta^4 U^3 + 3a^2\kappa^5\beta^7 U^5 + \kappa\beta^2\nabla^2 U + \kappa\beta^3\nabla^4 U - 2a\kappa^3\beta^5\nabla^2 U^3 + \mathcal{O}(\epsilon^{7/2}). \tag{36}$$

Substituting this expression into equation (28*a*), one can get an equation for the correction $\mathcal{G} = \mathcal{G}(\xi)$ to the soliton solution (34). The result is

$$\mathcal{L}_1\mathcal{G} \equiv \left(1 - \nabla_{\vec{\xi}}^2 - 3R^2\right)\mathcal{G} = \mathcal{L}_2 R, \tag{37}$$

with the operator $\mathcal{L}_2$ defined as follows:

$$\mathcal{L}_2 \equiv \left(1 + \frac{\kappa^2}{\mu_2^2}\right)^{-1} \frac{\kappa^2}{\mu_2^3} \left\{1 - 2\left(1 + \frac{a\kappa^2}{\mu_2^2}A_0^2\right)\nabla_{\vec{\xi}}^2 + \left(1 + 2\frac{a\kappa^2}{\mu_2^2}A_0^2\right)\nabla_{\vec{\xi}}^4\right\}. \tag{38}$$

Now, we can find the total number of atoms $N = N_u + N_v$. We have

$$N_u = A_0^2 N_{\mathrm{th}} + 2\epsilon A_0^2 \int \mathrm{d}^2\vec{\xi}\, R\mathcal{G} + \mathcal{O}(\epsilon^2). \tag{39}$$

The v-component of the soliton up to the order $\epsilon^{3/2}$ follows from equations (34) and (36):

$$V = \sqrt{\epsilon} A_0 \frac{\kappa}{\mu_2}\left\{R + \epsilon\left(\frac{1}{\mu_2}\left(\nabla_{\vec{\xi}}^2 - 1\right)R - A_0^2\frac{a\kappa^2}{\mu_2^3}R^3 + \mathcal{G}\right) + \mathcal{O}(\epsilon^2)\right\}. \tag{40}$$

Thus, the number of atoms N_v is given as

$$N_v = A_0^2\frac{\kappa^2}{\mu_2^2}N_{\mathrm{th}} + 2\epsilon\frac{\kappa^2}{\mu_2^2}A_0^2\mathcal{I} + \mathcal{O}(\epsilon^2), \tag{41}$$

where we have denoted

$$\mathcal{I} \equiv \int \mathrm{d}^2\vec{\xi} R\left(\frac{1}{\mu_2}\left(\nabla_{\vec{\xi}}^2 - 1\right)R - A_0^2\frac{a\kappa^2}{\mu_2^3}R^3 + \mathcal{G}\right) = \int \mathrm{d}^2\vec{\xi}\left\{R\mathcal{G} - \frac{2}{\mu_2}\left(1 + \frac{a\kappa^2}{\mu_2^2}A_0^2\right)R^2\right\}.$$

Relation (21) for the Townes soliton has been used to simplify the above expression for $\mathcal{I}$.

The scalar product of R and $\mathcal{G}$, which enters the expression for the number of atoms, is, in fact, equal to zero. Indeed, using equation (37), we get

$$\int \mathrm{d}^2\vec{\xi}\, R\mathcal{G} = \int \mathrm{d}^2\vec{\xi}\, R\mathcal{L}_1^{-1}\mathcal{L}_2 R,$$

but

$$\mathcal{L}_1^{-1} R = -\frac{1}{2}(R + \vec{\xi}\nabla_{\vec{\xi}} R). \tag{42}$$

(This equation can be obtained by taking the derivative of the stationary 2D NLS equation with respect to the chemical potential μ at $\mu = -1$.) Therefore,

$$\int \mathrm{d}^2\vec{\xi}\, R\mathcal{G} = -\frac{1}{2}\int \mathrm{d}^2\vec{\xi}\left(R\mathcal{L}_2 R + \frac{1}{2}\vec{\xi}\nabla_{\vec{\xi}}(R\mathcal{L}_2 R)\right) = 0,$$

via integration by parts in the second term. Hence, the total number of atoms assumes the following form:

$$N = N_u + N_v = \left(1 + \frac{\kappa^2}{\mu_2^2}\right) A_0^2 N_{\mathrm{th}} - 4\epsilon A_0^2 \frac{\kappa^2}{\mu_2^3}\left(1 + \frac{a\kappa^2}{\mu_2^2} A_0^2\right) N_{\mathrm{th}} + \mathcal{O}(\epsilon^2). \tag{43}$$

Clearly, (for $a \geqslant 0$) in the vicinity of the bifurcation point $\mu = \mu_1$, we have $\frac{\partial N}{\partial \mu} = -\frac{\partial N}{\partial \epsilon} > 0$ which renders the two-dimensional bifurcating solitons unstable in contrast to the stability of the similar bifurcating solitons in one spatial dimension [19].

4.2. Bifurcation from zero in the presence of a transverse potential

We have seen that the soliton bifurcation from zero is always discontinuous due to the fact that the solution has a new length scale $\xi = \sqrt{\epsilon}\rho$. If the transverse parabolic potential is taken into account (i.e. when it is not flat), the bifurcating solution has the length scale of order 1 (i.e. the order of the oscillator length). The stationary coupled-mode system with the parabolic transverse potential can be cast in a form analogous to that of system (28):

$$\omega U + \mathcal{D}U + U^3 + \kappa V = 0, \tag{44a}$$

$$(\omega - \varepsilon)V + \mathcal{D}V - aV^3 + \kappa U = 0, \tag{44b}$$

where we have introduced the operator $\mathcal{D} = \nabla^2 + 2 - \rho^2$ and a shifted chemical potential $\omega = \mu - 2$. Note that $\mathcal{D} \leqslant 0$ with $\mathcal{D}\,\mathrm{e}^{-\rho^2/2} = 0$. System (44) in the limit of a vanishing-amplitude solution, $U = A\,\mathrm{e}^{-\rho^2/2}$ and $V = B\,\mathrm{e}^{-\rho^2/2}$, with $A, B \to 0$, gives two boundary values of ω which coincide with the limiting chemical potentials of section 4.1: $\omega_{1,2} = \mu_{1,2}$. Moreover, the amplitudes are related as follows: $B_{1,2} = -(\omega_{1,2}/\kappa)A_{1,2}$. Therefore, the positive solution $AB > 0$ bifurcates at $\omega = \omega_1$, exactly as in the soliton case.

To study the bifurcation in detail let us set $\omega = \omega_1 - \epsilon$ with $\epsilon \to 0$ (here ϵ can be negative). Further steps in the derivation of the leading order equation for the bifurcating solution are formally the same as those in section 4.1, one has only to substitute $\mu \to \omega$ and $\nabla^2 \to \mathcal{D}$. For instance, we have $U \sim V \sim |\epsilon|^p$, where $p > 0$. For V we obtain the expression formally equivalent to that of equation (31),

$$V = \kappa\beta U - a\kappa^3\beta^4 U^3 + \kappa\beta^2\mathcal{D}U + 3a^2\kappa^5\beta^7 U^5 + \mathcal{O}(|\epsilon|^s), \tag{45}$$

with $\beta = (\varepsilon - \omega)^{-1}$ and $s = \min\{3p + 1, p + 2, 7p\}$, while U satisfies the equation

$$-\epsilon\left(1 + \frac{\kappa^2}{\mu_2^2}\right)U + \left(1 + \frac{\kappa^2}{\mu_2^2}\right)\mathcal{D}U + \left(1 - \frac{a\kappa^4}{\mu_2^4}\right)U^3 + \frac{3a^2\kappa^6}{\mu_2^7}U^5 = \mathcal{O}(|\epsilon|^s). \tag{46}$$

In the derivation of equations (45) and (46), it is assumed that $\mathcal{D} \sim \epsilon$, an analogue of $\nabla^2 \sim \epsilon$ of the previous subsection, though the reason is different: the operator $\mathcal{D}$ is small because of its discrete spectrum and the fact that the solution bifurcates from the ground state (with zero eigenvalue). The critical zero-point energy difference $\varepsilon_{\mathrm{cr}}$ (33) delineates the regions of the defocusing and focusing cases; though for $\varepsilon < \varepsilon_{\rm cr}$ the effective equation (46) is defocusing, it has a localized solution thanks to the external potential (in the operator $\mathcal{D}$).

To make our approach general we will use only two properties of the operator $\mathcal{D}$, namely that it is non-positive (with zero being an eigenvalue) and that it has a discrete spectrum. The case of $\varepsilon = \varepsilon_{\rm cr}$ is a special case of the bifurcation from zero, exactly as for the soliton bifurcation of the previous subsection. Consider first $\varepsilon \neq \varepsilon_{\rm cr}$, i.e. when the cubic term in equation (46) has a non-zero coefficient (positive or negative). Hence, $p = 1/2$ in this case. Defining a new dependent variable F by setting $U = \sqrt{|\epsilon|}|A_0|F(\rho)$, with A_0 given by equation (34), we obtain for F the equation

$$-F + \epsilon^{-1}\mathcal{D}F + \sigma F^3 = \mathcal{O}(|\epsilon|^{3/2}), \qquad \sigma = \operatorname{sgn}\{\epsilon(\varepsilon - \varepsilon_{\rm cr})\}, \tag{47}$$

which has a non-zero solution in the leading order ϵ^0 only if $\sigma > 0$, i.e. when the sign of ϵ is equal to that of $\varepsilon - \varepsilon_{\rm cr}$. Equation (47) allows one to obtain two leading orders of F in the expansion with respect to ϵ: $F = F_0 + \epsilon F_1 + \mathcal{O}(\epsilon^2)$. The simplest way to get them is to invert the operator $1 - \epsilon^{-1}\mathcal{D}$:

$$(1 - \epsilon^{-1}\mathcal{D})^{-1} = |\phi_0\rangle\langle\phi_0| + \epsilon\sum_{n=1}^{\infty}\frac{|\phi_n\rangle\langle\phi_n|}{\lambda_n + \epsilon}, \tag{48}$$

where we have used the eigenvalues, $-\lambda_n$ ($\lambda_n > 0$ for $n \geqslant 1$), and the eigenfunctions, $|\phi_n\rangle$, of the operator $\mathcal{D}$: $\mathcal{D}|\phi_n\rangle = -\lambda_n|\phi_n\rangle$. Now, multiplying equation (47) by the operator from equation (48) and collecting the successive orders of ϵ, we obtain

$$F = \langle\phi_0^4\rangle^{-1/2}\left\{1 - \frac{3\epsilon}{2\langle\phi_0^4\rangle^2}\sum_{n=1}^{\infty}\frac{\langle\phi_0^3\phi_n\rangle^2}{\lambda_n}\right\}\phi_0(\rho) + \epsilon\langle\phi_0^4\rangle^{-3/2}\sum_{n=1}^{\infty}\frac{\langle\phi_0^3\phi_n\rangle^2}{\lambda_n}\phi_n(\rho) + \mathcal{O}(\epsilon^2). \tag{49}$$

Here, $\langle\cdot\rangle$ denotes the integral $\int {\rm d}^2\vec{\rho}(\cdot)$.

Let us find the number of atoms corresponding to the bifurcating solution. We need just the leading order, since the bifurcation is continuous and the number of atoms tends to zero as $\omega \to \omega_1$. We get

$$N = \int {\rm d}^2\vec{\rho}(U^2 + V^2) = |\epsilon|\langle\phi_0^4\rangle^{-1}\left(1 + \frac{\kappa^2}{\mu_2^2}\right)A_0^2 + \mathcal{O}(\epsilon^2), \tag{50}$$

where we have used that in the leading order $V = \frac{\kappa}{\mu_2}U + \mathcal{O}(|\epsilon|^{3/2})$. Now, for $\epsilon > 0$, we have $\frac{\partial N}{\partial \mu} = -\frac{\partial N}{\partial \epsilon} < 0$, i.e. the solution is stable for $\varepsilon > \varepsilon_{\rm cr}$ by the VK criterion (since there is only one negative eigenvalue of the operator Λ_1 (24)). In the case $\varepsilon < \varepsilon_{\rm cr}$, the solution is unconditionally stable (the operator Λ_1 is positive). Unconditional stability in the latter case can be explained by the fact that the effective equation (46) is the defocusing NLS equation with the external potential, hence the ground-state solution is unconditionally stable.

Consider now the special case of the bifurcation from zero, when $\varepsilon = \varepsilon_{\rm cr}$. The leading order nonlinearity is quintic in U and we get $p = 1/4$. Defining a new variable F by setting $U = |\epsilon|^{1/4}B_0F(\rho)$, where $B_0^4 = \left(1 + \frac{\kappa^2}{\mu_2^2}\right)\frac{\mu_2^7}{3a^2\kappa^6}$, we get the effective equation

$$-F + \epsilon^{-1}\mathcal{D}F + \operatorname{sgn}(\epsilon)F^5 = \mathcal{O}(|\epsilon|^{3/4}), \tag{51}$$

which allows one to obtain the leading order $F = F_0 + \mathcal{O}(\epsilon)$. It is clear that in this case $\epsilon > 0$. We get $F_0 = \langle\phi_0^6\rangle^{-1/4}\phi_0(\rho)$. Finally, using $V = \frac{\kappa}{\mu_2}U + \mathcal{O}(\epsilon^{3/4})$, we obtain

$$N = N_u + N_v = \sqrt{\epsilon}\langle\phi_0^6\rangle^{-1/2}\left(1 + \frac{\kappa^2}{\mu_2^2}\right)B_0^2 + \mathcal{O}(\epsilon). \tag{52}$$

Therefore, in this case, the solution is also stable by the VK criterion (there is only one negative eigenvalue of the operator Λ_1 (24), similar to the previous case).

We see that the confining transverse potential allows for stable small-amplitude solutions. Moreover, for the zero-point energy difference below the critical value, $\varepsilon < \varepsilon_{\mathrm{cr}}$, the effective equation for the bifurcating solution is the defocusing NLS equation, whereas in the opposite case it is the focusing cubic or quintic NLS equation.

4.3. Asymptotic solution of the coupled-mode system for $\mu \to -\infty$

In the previous two subsections we have considered the bifurcation from zero. To complete the consideration, one has to study also the other limit of the chemical potential, i.e. $\mu \to -\infty$. This can be done in a unified way for the coupled-mode system with or without the transverse potential. The reason is that in both cases there is the same new length scale and the solution is approximated by the Townes soliton. Therefore, we will use the stationary system (44) to study the asymptotic solution. Thus, we set $\omega = -\epsilon^{-1}$ with $\epsilon \to 0$, where $\omega = \mu - 2$ as in the previous subsection. Finally, we will restrict the consideration to the case $a \geqslant 0$, which is the most interesting one, and assume that the transverse potential is parabolic (which is not an essential requirement, but is convenient for calculations).

Analysis of system (44) reveals that in the limit $\omega \to -\infty$, the leading order of the solution is as follows: $U = \mathcal{O}(\epsilon^{-1/2})$ and $V = \mathcal{O}(\epsilon^{1/2})$. Indeed, the leading order of V cannot be greater than that of U, otherwise there is no localized solution (for $a \geqslant 0$) in the limit $\epsilon \to 0$. Equation (44*a*) for U has a non-trivial solution only if $\mathcal{D} = \mathcal{O}(\epsilon^{-1})$. The only way to satisfy the latter is to require that $\nabla^2 \sim \epsilon^{-1}$, i.e. there is a new length scale $\xi = \epsilon^{-1/2}\rho$ (the effective length of the solution tends to zero as $\omega \to -\infty$ and the external potential is negligible). We will need to compute U up to the order $\epsilon^{3/2}$ and V up to the order $\epsilon^{1/2}$. Setting

$$U = \epsilon^{-1/2}(U^{(0)}(\epsilon^{-1/2}\rho) + \epsilon U^{(1)}(\epsilon^{-1/2}\rho) + \epsilon^2 U^{(2)}(\epsilon^{-1/2}\rho) + \mathcal{O}(\epsilon^3)),$$
$$V = \epsilon^{1/2}(V^{(0)}(\epsilon^{-1/2}\rho) + \mathcal{O}(\epsilon))$$

and expanding system (44) in the series with respect to ϵ, we obtain $U^{(0)} = R(\xi)$ and

$$\mathcal{L}_1 U^{(1)}(\xi) = 2R(\xi), \tag{53a}$$
$$\mathcal{L}_1 U^{(2)}(\xi) = 2U^{(1)}(\xi) - \xi^2 R(\xi) + 3R(\xi)U^{(1)}(\xi) + \kappa V^{(0)}(\xi), \tag{53b}$$
$$\left(1 - \nabla^2_{\vec{\xi}}\right) V^{(0)}(\xi) = \kappa R(\xi), \tag{53c}$$

where the operator $\mathcal{L}_1$ is given in equation (37). Since in the leading order we have the Townes soliton, ϵ is positive, i.e. the solutions with $\omega \to \infty$ are impossible (though for $\varepsilon < \varepsilon_{\mathrm{cr}}$, according to the results of section 4.2, the curve $N = N(\omega)$ corresponding to the solution of the coupled-mode system with the transverse potential initially enters the right half-plane $\omega > \omega_1$, it eventually turns left and approaches $-\infty$; see figure 5).

By setting $\kappa = 0$ in equation (53*b*), one must obtain the derivative $\frac{\partial N}{\partial \mu}$ corresponding to a single NLS equation (with or without the transverse potential) in the limit $\mu \to -\infty$. In the case without the transverse potential, we know that this derivative is zero due to the critical scale invariance. In the case of a single 2D NLS equation with an external potential, the derivative is negative and the number of atoms approaches the collapse threshold N_{th} (19) from below. 'Switching on' the quantum tunnelling changes this behaviour: for $\kappa > \kappa_{\mathrm{cr}}$, with some κ_{cr}, the derivative $\frac{\partial N}{\partial \mu}$ assumes a positive value and the number of atoms approaches the collapse threshold N_{th} from above. Let us find the critical tunnelling coefficient. We have

$$N_u = \int \mathrm{d}^2\vec{\rho}\, U^2 = N_{\mathrm{th}} + \epsilon^2 \int \mathrm{d}^2\vec{\xi}\{U^{(1)2} + 2RU^{(2)}\} + \mathcal{O}(\epsilon^3)$$
$$= N_{\mathrm{th}} - \epsilon^2 \int \mathrm{d}^2\vec{\xi}\{\vec{\xi}^2 R^2\} + \epsilon^2\kappa^2 \int \mathrm{d}^2\vec{\xi}\, U^{(1)}\left(1 - \nabla^2_{\vec{\xi}}\right)^{-1} R + \mathcal{O}(\epsilon^3),$$

$$N_v = \int \mathrm{d}^2\vec{\rho}\, V^2 = \epsilon^2\kappa^2 \int \mathrm{d}^2\vec{\xi}\, R\big(1 - \nabla_{\vec{\xi}}^2\big)^{-2} R + \mathcal{O}(\epsilon^3).$$

In the derivation of these formulae, we have solved equation (53*b*) for $U^{(2)}$ and (53*c*) for $V^{(0)}$ by the inversion of the corresponding operators and integration by parts. We have also used that the scalar product $\int \mathrm{d}^2\vec{\xi}\, RU^{(1)}$ is equal to zero, which is an immediate consequence of the fact that equation (53*a*) for $U^{(1)}$ does not involve the external potential (equation (42) also can be used to establish this fact directly). Using equation (53*a*) and integrating by parts to get rid of the term with $U^{(1)}$ in the last integral in the expression for N_u, we get the following formula for the total number of atoms:

$$N = N_u + N_v = N_{\mathrm{th}} - \epsilon^2 \int \mathrm{d}^2\vec{\xi}\, \xi^2 R^2 + \epsilon^2\kappa^2 \int \mathrm{d}^2\vec{\xi}\, R\big(1 - \nabla_{\vec{\xi}}^2\big)^{-1} R + \mathcal{O}(\epsilon^3). \tag{54}$$

For the coupled-mode system without the transverse potential, the first integral on the rhs of equation (54) is absent. The second integral is positive:

$$I_1 \equiv \int \mathrm{d}^2\vec{\xi}\, R\big(1 - \nabla_{\vec{\xi}}^2\big)^{-1} R \approx 7.41. \tag{55}$$

Hence, in this case, the number of atoms always approaches the threshold for collapse N_{th} of a single 2D NLS equation from above.

Consider now the case of the parabolic transverse potential. In this case, the second term on the rhs of equation (54) is negative and

$$I_2 \equiv \int \mathrm{d}^2\vec{\xi}\, \xi^2 R^2 \approx 13.82.$$

Therefore, the two terms of order ϵ^2 on the rhs of equation (54) compensate each other at some $\kappa = \kappa_{\mathrm{cr}}$ and the derivative $\frac{\partial N}{\partial \mu}$ changes sign in the limit $\mu \to -\infty$. The derivative itself is, however, of the order $-1/\omega^3 = \epsilon^3$ and the change of sign is not visible numerically.

The important conclusion from the asymptotic solution of the coupled-mode system for $\mu \to -\infty$ is that a new scale appears and the u-component of the solution is approximated by the Townes soliton, while the amplitude of the v-component tends to zero. Hence, there are always solutions which suffer from the collapse instability. They have the total number of atoms approaching N_{th} as $\mu \to -\infty$. The collapse instability also appears in this limit in the one-dimensional coupled-mode system [19], where for a large number of atoms the solution with $\mu \to -\infty$ is, in fact, the collapsing ground state. However, in the two-dimensional system with the transverse potential another stationary solution becomes the ground state (see the next section).

5. Numerical solution of the coupled-mode system

From the analytical study of the bifurcation from zero and the asymptotic solution for $\mu \to -\infty$, we can conclude the following. First of all, in the case of the coupled-mode system without the transverse potential, soliton solutions exist for $\mu \leqslant \mu_{\mathrm{bif}}$ with $\mu_{\mathrm{bif}} \equiv \mu_1$ from equation (29). The expansions of the total number of atoms for the bifurcating and asymptotic solitons are as follows:

$$N = \left(1 + \frac{\kappa^2}{\mu_2^2}\right) A_0^2 N_{\mathrm{th}} + 4(\mu - \mu_{\mathrm{bif}}) A_0^2 \frac{\kappa^2}{\mu_2^3} \left(1 + \frac{a\kappa^2}{\mu_2^2} A_0^2\right) N_{\mathrm{th}} + \mathcal{O}(\mu - \mu_{\mathrm{bif}})^2, \qquad \mu \to \mu_{\mathrm{bif}}, \tag{56}$$

$$N = N_{\mathrm{th}} + \frac{\kappa^2}{\mu^2} I_1 + \mathcal{O}(\mu^{-3}), \qquad \mu \to -\infty, \tag{57}$$

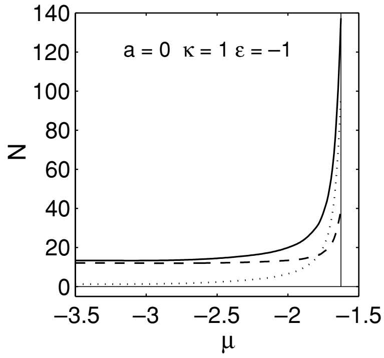

Figure 1. The total number of atoms (solid line) versus the chemical potential corresponding to the 2D soliton solutions of the coupled-mode system. The dashed and dotted lines give the number of atoms in the u- and v-condensates, respectively. The number of atoms here is reduced by the factor Δ, as in equation (17). The chemical potential is given in units of $\hbar\omega_\perp/2$.

where A_0 is defined in equation (34), $R(\xi)$ is the Townes soliton and I_1 is given by formula (55) (note that $\varepsilon > \varepsilon_{\mathrm{cr}}$ must be satisfied for the Townes soliton to exist). Hence, we have $N(\mu_{\mathrm{bif}}) > N_{\mathrm{th}}$ for all values of the system parameters (we consider $a \geqslant 0$). Moreover, in both limits, the derivative $\frac{\partial N}{\partial \mu}$ is positive. This gives a clear indication that the total number of atoms is a monotonic increasing function of the chemical potential and the soliton solutions to the system of linearly coupled focusing and defocusing 2D NLS equations are unstable in the whole domain of their existence. This conclusion was verified numerically. In figure 1 we show the number of atoms versus the chemical potential for the 2D solitons (in all figures we use the 'scaled number of atoms' N, related to the actual number of atoms by formula (17), and the dimensionless chemical potential which pertains to the coupled-mode system (15)).

For the coupled-mode system with the transverse potential, the behaviour of the number of atoms as a function of the chemical potential is much more interesting. In this case, the total number of atoms satisfies equations (50), (52) and (54). Generalizing these to an arbitrary confining potential, we have

$$N = \begin{cases} |\mu - \mu_{\mathrm{bif}}| \langle \phi_0^4 \rangle^{-1} \left(1 + \frac{\kappa^2}{\mu_2^2}\right)^2 \left|1 - \frac{a\kappa^4}{\mu_2^4}\right|^{-1} + \mathcal{O}(\mu - \mu_{\mathrm{bif}})^2, & \varepsilon \neq \varepsilon_{\mathrm{cr}}, \\ (\mu_{\mathrm{bif}} - \mu)^{1/2} \langle \phi_0^6 \rangle^{-1/2} \left[\frac{\mu_2^7}{3a^2\kappa^6}\left(1 + \frac{\kappa^2}{\mu_2^2}\right)^3\right]^{1/2} + \mathcal{O}(\mu - \mu_{\mathrm{bif}}), & \varepsilon = \varepsilon_{\mathrm{cr}}, \end{cases} \tag{58}$$

$$N = N_{\mathrm{th}} + \mu^{-2}(\kappa^2 I_1 + I_{\mathrm{ext}}) + \mathcal{O}(\mu^{-3}). \tag{59}$$

Here, $\langle \cdots \rangle = \int \mathrm{d}^2\vec{\rho}$ and $\phi_0(\rho)$ is the ground-state wavefunction of the linear operator $-\nabla^2 + V_{\mathrm{ext}}(\rho)$, where $V_{\mathrm{ext}}(\rho)$ (an even function) is the confining potential; I_1 is given by equation (55), while I_{ext} is determined by the quadratic term in the Taylor expansion of the trap $V_{\mathrm{ext}}(\rho)$ about $\rho = 0$ (for the parabolic trap $V_{\mathrm{ext}} = \rho^2$, we have $I_{\mathrm{ext}} \approx -13.82$). For the parabolic trap, $\langle \phi_0^4 \rangle^{-1} = 2\pi$ and $\langle \phi_0^6 \rangle^{-1} = 3\pi^2$.

As shown in the previous section, for $\mu \to -\infty$, the u-component of the solution tends to the Townes soliton, while the amplitude of the v-component tends to zero. Hence, the solution suffers from the collapse instability in this asymptotic limit. From equation (59), it is seen that the total number of atoms approaches N_{th}. However, one cannot conclude that the collapsing solution is the (unstable) ground state neither that there are no stable solutions to the coupled-mode system which have total number of atoms greater than N_{th}. In fact, such

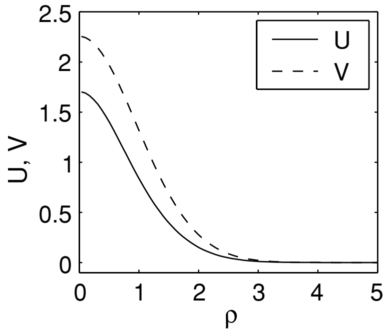

Figure 2. Ground state of the coupled-mode system (stable with respect to collapse). Here, the parameters are $a = 0.005$, $\kappa = 2$, $\varepsilon = -3$, $\mu = -2.31$ and $N = 21.5$. The order parameters U and V are given in units of $\sqrt{\Delta}/\ell_\perp$ and the radial length ρ in units of $\ell_\perp$.

solutions do exist and give the ground state of the system, *stable* with respect to collapse. Figure 2 illustrates an example of the stable ground-state solution with the total number of atoms larger than $N_{\rm th}$, there $a = 0.005$, $\kappa = 2$, $\varepsilon = -3$ and $N = 21.5$ (it corresponds to a point with $\mu = -2.31$ on the curve of figure 4). Indeed, equation (58) clearly states that the bifurcating solution is always stable (by the VK criterion for $\varepsilon \geqslant \varepsilon_{\rm cr}$ and unconditionally otherwise, according to the discussion in sections 3 and 4.2).

Let us show that the external parabolic potential can stabilize the Townes soliton (by transforming it into a localized solution with the radius fixed by the oscillator length $\ell_\perp$). To this end, we will use formulae (56) and (58), i.e. we consider the limit $\mu \to \mu_{\rm bif}$ (we also assume that $\varepsilon > \varepsilon_{\rm cr}$). The expansions for the number of atoms of the Townes-type soliton and the ground state in the parabolic trap can be written as follows: $N_{\rm TS} = C_0 + C_1(\mu - \mu_{\rm bif}) + \mathcal{O}(\mu - \mu_{\rm bif})^2$ and $N_{\rm GS} = D_1|\mu - \mu_{\rm bif}| + \mathcal{O}(\mu - \mu_{\rm bif})^2$. Note that all coefficients are positive. The condensate with $N = N_{\rm TS}$, i.e. which corresponds to unstable Tonwes-type soliton close to the bifurcation point, is stabilized by the parabolic potential if $N_{\rm GS} \geqslant N_{\rm TS}$. We obtain $D_1 + C_1 \gg C_0$ (since $\mu < \mu_{\rm bif}$ and we consider the limit $\mu \to \mu_{\rm bif}$). This inequality can be resolved easily for $\kappa \geqslant |\varepsilon|$. In this case, $\mu_2 \sim \kappa$. Using equations (56) and (58), the definition of A_0 and the value of $N_{\rm th}$, we obtain the condition for the condensate stabilization $\kappa \ll (1 + 2a/(1 - a))^{-1}$. On the other hand, the applicability condition (18) requires that $\gamma \gg \max(N_u, aN_v)$. For sufficiently small $a \ll 1$, the maximum is given by $N_u \sim N_{\rm th} = 11.69$. Thus, $\gamma \gg 10$. The number of atoms in the stabilized condensate with the chemical potential $\mu \to \mu_{\rm bif}$ can be estimated as $\mathcal{N} \sim 4N_{\rm th}\Delta = 47\Delta$, where usually $\Delta \sim 10^2$–10^3. The actual value of the number of atoms in a stable stationary state could be much larger than the estimate above, as is evident from figure 6. (Note also that the above restriction on the values of κ can in fact be violated as is manifested in figures 3 and 4.)

To determine the ground state of the system, we have calculated the energy of the solution numerically. In the coupled-mode approximation, the energy of the condensate in the double-well trap is given as

$$E = \frac{\hbar\omega_\perp \left(\int {\rm d}z|\psi_u|^4\right)^{-1}}{16\pi\left|a_{\rm s}^{(u)}\right|}\mathcal{H},$$
$$\mathcal{H} = \int {\rm d}^2\vec{\rho}\left\{|\nabla u|^2 + |\nabla v|^2 + \rho^2(|u|^2 + |v|^2) + \varepsilon|v|^2 - \kappa(uv^* + vu^*) - \frac{|u|^4}{2} + a\frac{|v|^4}{2}\right\}. \qquad (60)$$

Here, $\mathcal{H}$ is the Hamiltonian of the dimensionless coupled-mode system.

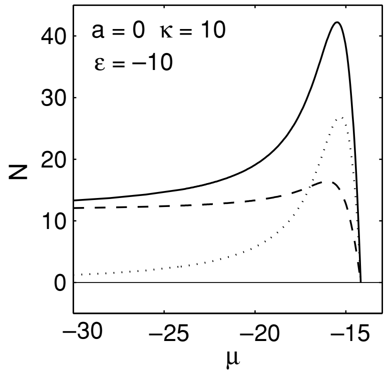

Figure 3. The total number of atoms versus the chemical potential (the solid line) for $a = 0$. The dashed and dotted lines give the number of atoms in the u- and v-condensates, respectively. The axes units are as in figure 1.

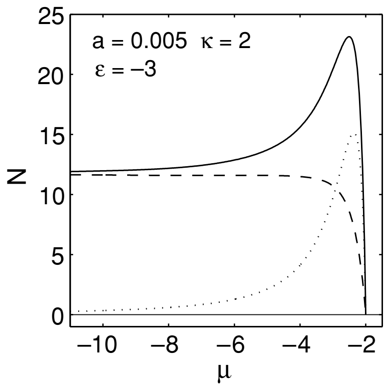

Figure 4. The total number of atoms versus the chemical potential (the solid line) for $a = 0.005$. Here, $\varepsilon > \varepsilon_{\mathrm{cr}}$. The dashed and dotted lines give the number of atoms in the u- and v-condensates, respectively. The axes units are as in figure 1.

The existence of stable stationary solutions with large total number of atoms depends on the system parameters, principally on the tunnelling coefficient κ and the zero-point energy difference ε. In the asymptotic limit of weakly coupled equations, i.e. when $\mu \to -\infty$, the ground state is unstable with respect to collapse, similar to a single NLS equation with external potential. For large values of κ and large negative ε, one can expect the appearance of a new ground state due to strong quantum tunnelling through the barrier and competition of the attraction in the u-condensate and the lower zero-point energy for the atoms in the v-condensate. Stable solutions with a large total number of atoms were indeed found, for instance, for $a = 0$, $\kappa = 10$ and $\varepsilon = -10$; the corresponding dependence of the total number of atoms on the chemical potential is given in figure 3. We have checked numerically (by a numerical analysis of the spectrum of Λ_1 (24) from section 3) that the VK criterion applies. Therefore, the stable solutions pertain to the region where $\frac{\partial N}{\partial \mu} < 0$, close to the point of the bifurcation from zero.

Figure 3 corresponds to the special case of $a = 0$, i.e. when the v-condensate is a non-interacting quantum gas. However, for $a > 0$, when the v-condensate is repulsive, the curve $N = N(\mu)$ exhibits similar behaviour, see figure 4.

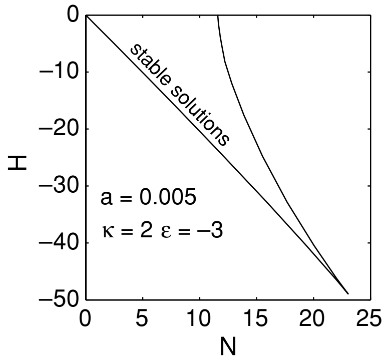

Figure 5. The energy of solutions versus the total number of atoms, corresponding to figure 4. The almost straight line contains the ground state of the system and corresponds to the part of the curve $N = N(\mu)$ in figure 4 where $\frac{\partial N}{\partial \mu} < 0$. Here, the number of atoms is reduced by the factor Δ from equation (17), whereas the energy is given in units of $\hbar\omega_\perp\Delta/2$.

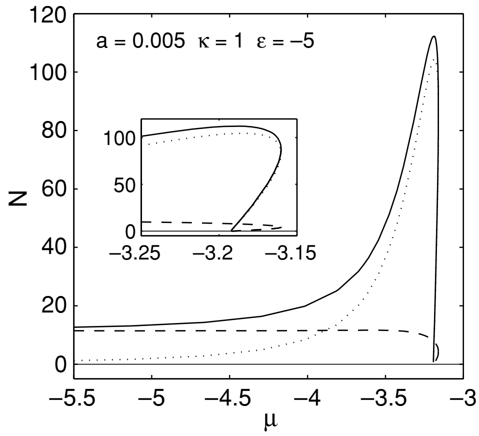

Figure 6. The total number of atoms versus the chemical potential (the solid line). The dashed and dotted lines give the number of atoms in the u- and v-condensates, respectively. Here, $\varepsilon < \varepsilon_{cr}$. The inset shows a section of the figure about the bifurcation point. The axes units are as in figure 1.

The part of the curve $N = N(\mu)$ with $\frac{\partial N}{\partial \mu} < 0$, corresponding to the stable solutions with a large number of atoms, also minimizes the energy when there is a local maximum with the total number of atoms higher than N_{th}. This is illustrated in figure 5, where we give the equation of state, $\mathcal{H} = \mathcal{H}(N)$ (i.e. the energy versus the total number of atoms in the dimensionless variables). The equation of state given in this figure is characteristic for the 2D coupled-mode system for a large region of values of the system parameters, when there is a maximum of the total number of atoms higher than N_{th}. Otherwise, the energy takes positive values and the minimum (zero) corresponds to the collapsing ground state.

Figures 3 and 4 correspond to the case $\varepsilon > \varepsilon_{cr}$. A new feature appears in the opposite case, i.e. when $\varepsilon < \varepsilon_{cr}$. Indeed, in this case, the effective equation (see equation (46)) is defocusing close to the bifurcation point. This fact changes drastically the dependence of the total number of atoms on chemical potential: the curve $N = N(\mu)$ enters into the region of $\mu > \mu_{bif}$, i.e. to the right of the bifurcation point, see the inset in figure 6.

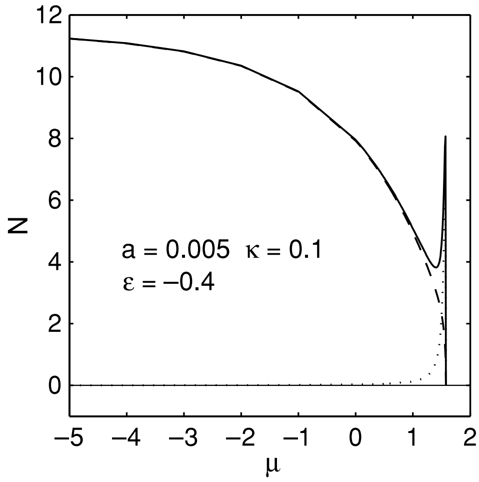

Figure 7. The total number of atoms versus the chemical potential (the solid line) for small values of the tunnelling coefficient and energy difference. Here, $\varepsilon < \varepsilon_{cr}$. The dashed and dotted lines give the number of atoms in the u- and v-condensates, respectively. The axes units are as in figure 1.

The protruding part of the curve is analogous to that in a single defocusing 2D NLS equation with an external potential. The operator Λ_1 appearing in the linear stability analysis of section 3 is positive definite there. Hence, the solutions corresponding to this part of the curve $N = N(\mu)$ are unconditionally stable.

Moreover, there is a turning point bifurcation, where $\frac{\partial N}{\partial \mu} = \infty$ (see the inset in figure 6). This bifurcation corresponds to the change of sign of the effective nonlinearity in the coupled-mode system from negative to positive as one moves upwards along the curve starting from the bifurcation point. Accordingly, the lowest positive eigenvalue of the operator $\Lambda_{1,0}$ passes through zero and becomes negative to the left of the turning point. As there are no other negative eigenvalues, the VK criterion applies to the left of the turning point bifurcation.

The equation of state $\mathcal{H} = \mathcal{H}(N)$ corresponding to figure 6 is similar to that illustrated in figure 5. Accordingly, the total number of atoms in the ground state exceeds $N_{\rm th}$ by an order of magnitude in this case.

Finally, figure 7 illustrates the fact that the maximum of the total number of atoms achievable by the stationary solution is determined by the tunnelling coefficient κ and the zero-point energy difference ε (compare to figure 3). In this case, $\varepsilon < \varepsilon_{cr}$ (the turning point bifurcation as well as the protruding part of the curve are also present, but not visible). The stability is again determined by the VK criterion, except for the extremely narrow region before the turning point bifurcation (where the solution is unconditionally stable). In this case, the total number of atoms is always smaller than the threshold $N_{\rm th}$.

As $\mu \to -\infty$, the curves $N = N(\mu)$ and $N_{u,v} = N_{u,v}(\mu)$ in figures 3, 4, 6 and 7 are similar to that of the 'solitonic' curve shown in figure 1. We have confirmed that the asymptotic solution for $\mu \to -\infty$ indeed approaches the Townes soliton in its u-component, whereas the v-component tends to zero. Thus, in the limit $\mu \to -\infty$, the two condensates are weakly coupled, similar to the one-dimensional case [19].

The stable ground state in the 2D coupled-mode system, which corresponds to the part of the curve $N = N(\mu)$ immediately after the bifurcation point, appears due to breaking of the scale invariance by the transverse trap. This is reflected in the estimate of the solution width which is of the order of the oscillator length of the trap (see section 4.2). It is seen that the larger share of atoms is gathered in the repulsive v-condensate. Hence, such a state is similar to the unusual bright soliton of the one-dimensional coupled-mode system [19]. However, there is an important difference between the one- and two-dimensional cases: in one spatial dimension the ground state always corresponds to the weakly coupled condensates and suffers from the collapse instability for a large number of atoms.

6. Conclusion

Nonlinearity of the Gross–Pitaevskii equation is due to the atomic interaction which is essential for understanding the properties of the condensate. Control over the nonlinearity in the Gross–Pitaevskii equation allows for coupling of two condensates with the nonlinearities of opposite signs (the scattering lengths). This can be realized using a double-well trap with far-separated wells. In the one-dimensional case, there are stable bright solitons with almost all atoms gathered in the repulsive condensate [19]. In the two-dimensional case, on the other hand, the Townes-type solitons in the system are always unstable due to the fact that the vanishing-amplitude 2D soliton solution has a finite l_2-norm, i.e. the bifurcation from zero always corresponds to a discontinuity in the dependence of the number of atoms on the chemical potential. This is quite dissimilar to the one-dimensional coupled-mode system, where the bifurcation from zero is continuous except for the boundary case [19].

With the use of a parabolic potential, the spatial scale of the solution is fixed by the oscillator length. This allows for the stable stationary solutions (though they are not solitons) with large l_2-norms, which represent the ground state of the system. The ground state is secured from the collapse instability by an energy barrier. Interestingly, this ground-state solution may have the l_2-norm, i.e. the number of atoms of the condensate in a double-well trap, much higher than the collapse threshold in a single 2D NLS equation (in some cases, the total number of atoms exceeds the collapse threshold by an order of magnitude). This is a new phenomenon, which pertains only to the two-dimensional coupled-mode system, since in the one-dimensional case, for a large number of atoms, the ground state corresponds to weakly coupled condensates, has a large negative chemical potential and suffers from the collapse instability [19]. A more detailed study of the properties of the energy barrier for collapse is relegated to [21].

Recently, a system of equations similar to the coupled-mode system was elucidated in the context of two parallel coupled BECs [39]. Depending on the coupling, the system exhibits transition from a dark soliton to the Josephson vortex and vice versa. This opens a new direction for research related to the coupled-mode system.

Acknowledgments

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